Introduction to Spectral Theory

Second lecture: The spectral theorem

Ingo Witt (Göttingen)

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Overview

In this lecture, we will state the spectral theorem for bounded self-adjoint operators and sketch a proof. We closely follow Chapter VII of M. Reed and B. Simon *Methods of Modern Mathematical Physics*.

Here are the single steps:

1. Continuous functional calculus
2. Spectral theorem: Functional calculus form (bounded functional calculus)
3. Spectral theorem: Multiplication operator form
4. Spectral theorem: Projection-valued measure form

Continuous functional calculus

Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then there exists a unique linear map $\Phi\colon \mathscr{C}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:

(a) Φ *is a* $*$*-homomorphism, i.e.,*

$$\Phi(fg) = \Phi(f)\Phi(g), \quad \Phi(\bar{f}) = \Phi(f)^*, \quad \Phi(1) = \mathrm{I}.$$

(b) Φ *is norm continuous.*

(c) $\Phi(\lambda \mapsto \lambda) = A$*.*

Moreover, Φ *has the additional properties*:

(d) *If* $A\varphi = \lambda\varphi$*, then* $\Phi(f)\varphi = f(\lambda)\varphi$*.*

(e) $\sigma(\Phi(f)) = \{f(\lambda) \mid \lambda \in \sigma(A)\}$ (spectral mapping theorem).

(f) *If* $f \geq 0$*, then* $\Phi(f) \geq 0$*.*

(g) $\|\Phi(f)\| = \|f\|_\infty$*.*

Sketch of proof

By (a) and (c), if f is a polynomial, then $\Phi(f) = f(A)$.

It suffices to prove that (g) holds when f is a polynomial, i.e.,

(*) $$\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|.$$

By the Weierstrass approximation theorem, Φ can then be uniquely extended from the set of polynomials to all of $\mathscr{C}(\sigma(A))$.

Notation Later, we will write $f(A)$ in place of $\Phi(f)$ in general.

Proof of (*)

We will show that

$$\sigma(f(A)) = \{f(\lambda) \mid \lambda \in \sigma(A)\}.$$

Then (*) follows, since $f(A)$ is normal.

1. Let $\lambda \in \sigma(A)$. Write $f(\mu) - f(\lambda) = (\mu - \lambda)\, g(\mu)$, where g is a polynomial. Then $f(A) - f(\lambda) = (A - \lambda)\, g(A)$. We infer that $f(\lambda) \in \sigma(f(A))$, since $A - \lambda$ is not boundedly invertible.
2. Conversely, let $\kappa \in \sigma(f(A))$. Write $f(\mu) - \kappa = a\,(\mu - \lambda_1)\ldots(\mu - \lambda_k)$, where $a \neq 0$ and $k = \deg f$. If $\lambda_1, \ldots, \lambda_k \notin \sigma(A)$, then

   $$f(A) - \kappa = a\,(A - \lambda_1)\ldots(A - \lambda_k)$$

   is boundedly invertible, which contradicts $\kappa \in \sigma(f(A))$. Therefore, $\lambda_j \in \sigma(A)$ for some j, while $\kappa = f(\lambda_j)$. $\square$

Two remarks

1. The range of Φ,

$$\operatorname{ran}\Phi = \{f(A) \mid f \in \mathscr{C}(\sigma(A))\},$$

is the C^*-algebra generated by A.

Indeed, $\operatorname{ran}\Phi$ is $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ by functional calculus, and it is norm closed in $\mathcal{L}(\mathcal{H})$ as $\|f(A)\| = \|f\|_\infty$ and $\mathscr{C}(\sigma(A))$ is complete.

2. $f(M_g) = M_{f\circ g}$ for $g \in L^\infty(X,\mu)$, $f \in \mathscr{C}(\operatorname{ran} g)$.

Spectral theorem

Functional calculus form

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint, $\psi \in \mathcal{H}$. By the Riesz-Markov representation theorem, there exists a unique finite Borel measure $d\mu_\psi$ on $\sigma(A)$ such that

$$\langle f(A)\psi, \psi \rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_\psi.$$

This allows to extend the functional calculus to include bounded Borel functions $f \colon \sigma(A) \to \mathbb{C}$.

Theorem

There is a unique linear map Φ from the bounded Borel functions on $\sigma(A)$ to $\mathcal{L}(\mathcal{H})$ such that the following holds:

(a) *Φ is a $*$-homomorphism,*

(b) *Φ is norm continuous,*

(c) *$\Phi(\lambda \mapsto \lambda) = A$,*

(d) *If $f_n \to f$ pointwise and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n) \to \Phi(f)$ strongly.*

Spectral theorem

Functional calculus form, II

Theorem (cont.)

Moreover, Φ has the additional properties:

(e) *If $A\varphi = \lambda\varphi$, then $\Phi(f)\varphi = f(\lambda)\varphi$.*

(f) *If $f \geq 0$, then $\Phi(f) \geq 0$.*

(g) *If $B \in \mathcal{L}(\mathcal{H})$ commutes with A, then B commutes with $\Phi(f)$.*

Again, we write $f(A)$ in place of $\Phi(f)$.

Remark Now, the set of $f(A)$, where f runs through all bounded Borel functions on $\sigma(A)$, is the von Neumann algebra generated by A.

Remark To get a statement like $\|f(A)\| = \|f\|_\infty$, one needs an adequate notion of "almost everywhere." One possibility is to pick an orthonormal basis $\{\varphi_n\}$ of $\mathcal{H}$ and declare that a property holds a.e. if it holds a.e. with respect to each $d\mu_{\varphi_n}$.

Spectral theorem

Multiplication operator form

Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then A is unitarily equivalent to a multiplication operator, i.e., there are a measure space (X, μ), a real-valued function $g \in L^\infty(X, \mu)$, and a unitary operator $U \in \mathcal{L}(\mathcal{H}, L^2(X, \mu))$ such that

$$A = U^* M_g U.$$

Sketch of proof, I

Definition

$\psi \in \mathcal{H}$ is called a cyclic vector for A if the span of $\{A^n\psi \mid n \in \mathbb{N}_0\}$ is dense in $\mathcal{H}$.

Lemma

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint with cyclic vector ψ. Then A is unitarily equivalent to $M_{\lambda\mapsto\lambda}$ on $L^2(\sigma(A), d\mu_\psi)$.

Proof Define $U \in \mathcal{L}(\mathcal{H}, L^2(\sigma(A), d\mu_\psi))$ by $U(f(A)\psi) = f$ for $f \in \mathscr{C}(\sigma(A))$ and verify by direct computation that

$$\|f(A)\psi\|^2 = \int_{\sigma(A)} |f(\lambda)|^2 \, d\mu_\psi$$

as well as $(UAU^* f)(\lambda) = \lambda f(\lambda)$. $\square$

Sketch of proof, II

Recall that $\mathcal{H}$ is assumed to be separable.

Lemma

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then one can write

$$\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n,$$

where $N \in \mathbb{N} \cup \{\infty\}$, such that A leaves each $\mathcal{H}_n$ invariant and $A\restriction_{\mathcal{H}_n}$ possesses a cyclic vector.

Both lemmas together prove the spectral theorem in its multiplication operator form. $\square$

Spectral projections

Definition

An assignment $\mathscr{B}(\mathbb{R}) \ni \Omega \mapsto E_\Omega$, where $E_\Omega = E_\Omega^* = E_\Omega^2 \in \mathcal{L}(\mathcal{H})$ is said to be a projection-valued measure if the following conditions are met:

- $E_\emptyset = 0$, $E_{[-M,M]} = \mathrm{I}_\mathcal{H}$ for some $M > 0$,
- $E_\Omega E_{\Omega'} = E_{\Omega \cap \Omega'}$,
- If $\Omega_k \uparrow \Omega$, then $E_\Omega = s\text{-}\lim_{k\to\infty} E_{\Omega_k}$.

For $\lambda \in \mathbb{R}$, set $E_\lambda = E_{(-\infty,\lambda]}$. Then, for any $\psi \in \mathcal{H}$, $\mathrm{d}\langle E_\lambda \psi, \psi\rangle$ is a finite Borel measure, of total mass $\|\psi\|^2$. Hence, setting

$$\langle A\psi, \psi\rangle = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}\langle E_\lambda \psi, \psi\rangle$$

defines, by polarization, a linear bounded operator A on $\mathcal{H}$.

Spectral theorem

Projection-valued measure form

Theorem

There is a one-to-one correspondence between bounded self-adjoint operators A and projection-valued measures $\{E_\Omega\}_{\Omega \in \mathscr{B}(\mathbb{R})}$.

Remarks

1. $d\mu_\psi = d\langle E_\lambda \psi, \psi \rangle$.
2. $E_\Omega = \chi_\Omega(A)$.

Notation $\{E_\Omega\} = \{E_\Omega^A\}$ is called the spectral measure of A.

Sketch of proof

1. Theorem holds for multiplication operators.

Indeed, $E^g_\Omega = M_{\chi_{\{g\in\Omega\}}}$. Then (use the Riemann-Stieltjes integral)

$$\int_{-\infty}^{\infty} \lambda \, \mathrm{d}\langle E^g_\lambda \psi, \psi\rangle \approx \sum_{j=1}^{k} \mu_j \int_{\{\lambda_{j-1}<g\leq\lambda_j\}} |\psi|^2 \, \mathrm{d}\mu$$

$$\xrightarrow{\sup_j |\lambda_j - \lambda_{j-1}| \to 0} \int_X g\, |\psi|^2 \, \mathrm{d}\mu = \langle M_g \psi, \psi\rangle,$$

where $\lambda_0 < \lambda_1 < \ldots < \lambda_k$ is a partition of $\operatorname{ran} g$ and $\mu_j \in [\lambda_{j-1}, \lambda_j]$.

2. For arbitrary A, write $A = U^* M_g U$.

Then $E^A_\Omega = U^* E^g_\Omega U$ and

$$A = U^* M_g U = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}(U^* E^g_\lambda U) = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E^A_\lambda.$$

Integrals here are Lebesgue-Stieltjes integrals in the weak sense. $\square$

Further examples

1. Let A be an $N \times N$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and eigenvectors $\varphi_1, \ldots, \varphi_N$ chosen to form an orthonormal basis of $\mathbb{C}^N$. Regarding $A \in \mathcal{L}(\mathbb{C}^N)$ as self-adjoint operator on $\mathbb{C}^N$, we have that

$$E_\Omega = \sum_{j\colon \lambda_j \in \Omega} \langle \cdot, \varphi_j \rangle \varphi_j.$$

2. Let A be a compact self-adjoint operator with eigenvalues λ_j and eigenfunctions φ_j, again chosen to form an orthonormal basis. Then the same formula for E_Ω holds.

Back to the functional calculus

Given a bounded Borel function f, we recover $f(A)$ for a bounded self-adjoint operator A with spectral measure dE_λ as

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda,$$

where (again) the integral has to be understood as a Lebesgue-Stieltjes integral in the weak sense.

Classification of the spectrum

Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ *be self-adjoint. Then*

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}}(A) \oplus \mathcal{H}_{\mathrm{ac}}(A) \oplus \mathcal{H}_{\mathrm{sc}}(A),$$

where

$$\begin{aligned} \mathcal{H}_{\mathrm{pp}}(A) &= \{\psi \in \mathcal{H} \mid d\mu_\psi \text{ is supported in a countable set}\}, \\ \mathcal{H}_{\mathrm{ac}}(A) &= \{\psi \in \mathcal{H} \mid d\mu_\psi \ll d\lambda\}, \\ \mathcal{H}_{\mathrm{sc}}(A) &= \{\psi \in \mathcal{H} \mid d\mu_\psi \perp d\lambda, \text{ but with no point mass}\}. \end{aligned}$$

Here, pp - pure point, ac - absolutely continuous, sc - singularly continuous.

Proposition

Let $* \in \{\mathrm{pp}, \mathrm{ac}, \mathrm{sc}\}$. Then $\mathcal{H}_*(A)$ is invariant for A. Moreover, the spectrum of $A\restriction_{\mathcal{H}_*(A)}$ is purely $*$ in the sense that, for $\# \in \{\mathrm{pp}, \mathrm{ac}, \mathrm{sc}\} \setminus \{*\}$,

$$\mathcal{H}_\# \left(A\restriction_{\mathcal{H}_*(A)}\right) = \{0\}.$$

We write $\sigma_*(A) = \sigma\left(A\restriction_{\mathcal{H}_*(A)}\right)$. Note that

$$\sigma(A) = \sigma_{\mathrm{pp}}(A) \cup \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{sc}}(A),$$

but $\sigma_{\mathrm{pp}}(A)$, $\sigma_{\mathrm{ac}}(A)$, and $\sigma_{\mathrm{sc}}(A)$ need not be disjoint (as compact subsets of $\mathbb{R}$).